Direct sums

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Let $U_1, \ldots, U_k$ be a family of vector subspaces of a vector space V over a field $\mathbb{F}$.

Given any family of subspaces $U_1, \ldots, U_k$ we defined in Linear Algebra 1 its sum to be the smallest vector subspace of V containing all of the U_i's. Equivalently,

$$U_1 + \ldots + U_k = \{u_1 + \ldots + u_k \mid u_i \in U_i\}$$

Recall that in the particular case of two subspaces of a finite dimensional space we have the formula:

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

In particular by induction we have $\dim(U_1 + \ldots + U_k) \leq \dim(U_1) + \ldots + \dim(U_k)$.

Definition 1: *We say that a family $U_1, \ldots, U_k$ of vector subspaces of V is linearly independent if for any choice of vectors $u_1 \in U_1, \ldots, u_k \in U_k$, we have*

$$u_1 + \ldots + u_k = 0 \iff u_1 = 0, \ldots, u_k = 0$$

In that case, we call $U_1 + \ldots + U_k$ the **direct sum** *of the spaces $U_1, \ldots, U_k$, and we denote it by $U_1 \oplus \ldots \oplus U_k$.*

Remark 2: *This is equivalent to saying that any vector $u \in U_1 + \ldots + U_k$ has a unique representation as $u_1 + \ldots + u_k$ with $u_i \in U_i$ for each i: suppose $u = u_1 + \ldots + u_k = u'_1 + \ldots + u'k$, we deduce $(u_1 - u'_1) + \ldots + (u_k - u'_k) = 0$. Since $u_i - u'_i \in U_i$, we get that for each i $u_i = u'_i$ - the representation of u is unique. Conversely, if each vector has a unique representation, in particular so does the zero vector, hence the only way to write it as a sum $0 = u_1 + \ldots + u_r$ is to write $0 = 0 + \ldots + 0$.*

In particular for any $i \neq j$ we have $U_i \cap U_j = \{0\}$, but it is much stronger than this.

Example 3:

- In $\mathbb{R}^3$: two distinct lines in $\mathbb{R}^3$ are linearly independent. A line and a plane not containing it are linearly independent. Two planes are never linearly independent. Three distinct lines are linearly independent iff they are not contained in a common plane.
- A family of nonzero vectors $v_1, \ldots, v_k$ is linearly independent $\iff$ the corresponding family of subspaces $\mathrm{Span}(v_1), \ldots, \mathrm{Span}(v_k)$ is linearly independent.

Proposition 4: *Let $U_1, \ldots, U_k$ be a family of finite dimensional vector subspaces of a vector space V. The following are equivalent:*

1. *$U_1, \ldots, U_k$ are linearly independent;*
2. *for any i, $U_i \cap (\sum_{j \neq i} U_j) = \{0\}$*
3. *$\dim U = \dim(U_1) + \ldots + \dim(U_k)$.*

Proof. ($1 \implies 2$) Let $v \in U_i \cap \sum_{j \neq i} U_i$. Then $v \in U_i$, so $v = 0 + \ldots + 0 + v + 0 + \ldots + 0$ but we also have $v = u_1 + \ldots + u_{i-1} + u_{i+1} + \ldots + u_k$ for $u_j \in U_j$ for all $j \neq i$. By independence of the spaces, all the vectors are zero, hence so is V.

(2 $\implies$ 3) By induction on k. For $k=1$, this is clear. Induction: denote $V_k = U_1 + \ldots + U_k$, we have $\dim(V_k + U_{k+1}) = \dim(V_k) + \dim(U_{k+1}) - \dim(V_k \cap U_{k+1})$. By induction hypothesis, $\dim(V_k) = \dim(U_1) + \ldots + \dim(U_k)$, and by (2) we have $V_k \cap U_{k+1} = \{0\}$, so we get the result.

(3 $\implies$ 1) Note that applying the formula for the dimension of the sum of two subspaces we get

$$\dim(U_1 + \ldots + U_k) = \dim(U_1) + \ldots + \dim(U_k) - \sum_i \dim(U_i \cap \sum_{j=1}^{i-1} U_j)$$

Thus if 3 holds, we have for each i that $U_i \cap \sum_{i \neq j} U_j = \{0\}$. Suppose now $u_1 + \ldots + u_k = 0$, assume by contradiction that not all the u_i's are zero, and let i be maximal so that $u_i \neq 0$. We have $u_i = -(u_{i-1} + \ldots + u_1)$ thus $u_i \in U_i \cap \sum_{j=1}^{i-1} U_j$. This is a contradiction. Thus $u_i = 0$ for all i. $\square$

(The equivalence between (1) and (2) holds also for infinite dimensional vector spaces).

Proposition 5: *Let $U_1, \ldots, U_k$ be an independent family of finite dimensional vector subspaces. For each i, let $\mathcal{A}_i = \{v_1^i, \ldots, v_{l_i}^i\}$ be a linearly independent set of vectors in U_i. Then $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$ is linearly independent.*

Proof. Suppose that $(\lambda_1^1 v_1^1 + \ldots + \lambda_{l_1}^1 v_{l_1}^1) + \ldots + (\lambda_1^k v_1^k + \ldots + \lambda_{l_k}^k v_{l_k}^k) = 0$. Since $U_1, \ldots, U_k$ is an independent family of vector spaces, for each i we must have $\lambda_1^i v_1^i + \ldots + \lambda_{l_i}^i v_{l_i}^i = 0$. By linear independence of $v_1^i, \ldots, v_{l_i}^i$ we get $\lambda_1^i = \ldots = \lambda_{l_i}^i = 0$, which proves the result. $\square$

Corollary 6: *Let $U_1, \ldots, U_k$ be an independent family of finite dimensional vector subspaces. If $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are bases for $U_1, \ldots, U_k$ respectively, then $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k$ is a basis for $U_1 \oplus \ldots \oplus U_k$*

Proof. By the previous proposition, $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k$ is a linearly independent family of vectors. But its cardinality is $\sum_{i=1}^k \dim(U_i)$, which by Proposition 4 is exactly $\dim(U_1 \oplus \ldots \oplus U_k)$. $\square$